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1983 J. Phys. A: Math. Gen. 16 1

(http://iopscience.iop.org/0305-4470/16/1/010)

J. Phys. A: Math. Gen. 16 (1983) 1–7. Printed in Great Britain

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Received 20 April 1982

Abstract. Mathematical tools of modern differential geometry are used to derive, in an intrinsic formulation, more general results about non-Noether constants of motion. A relation between two different ways of obtaining such constants is found by making use of Leverrier's method of determining the characteristic polynomial of a matrix in terms of the traces of its increasing powers.

1. Introduction

The role played by symmetry principles in classical mechanics has been less important than in quantum mechanics, no doubt because the mathematical model describing the quantum systems was well established from its very beginning: Hilbert spaces, self-adjoint operators, and so on. Furthermore, the mathematical theory of linear representations of groups, which has been shown to be a very useful tool, was available for physicists. On the other hand, the mathematical framework of classical mechanics has not been well defined for a long time. The quantitative dynamics as developed by Lagrange, Hamilton, Jacobi, etc during the nineteenth century was unable to solve the oldest problems of celestial mechanics. Poincaré pointed out the necessity of introducing new methods and concepts, giving rise to the so-called qualitative dynamics which makes use of more advanced methods and tools of differential geometry to take into account global properties. This program could not be carried out until the end of the forties when Cartan developed the calculus of differentiable forms on manifolds. The recent development of geometric quantisation has compelled theoretical physicists to manage with a lot of geometrical concepts which can be shown to be very useful when appropriately used in classical mechanics. We aim to show in this paper how some very recent results on non-Noether constants of motion can be stated in a simpler way with an intrinsic formulation, allowing a straightforward generalisation to more general situations.

The paper is organised as follows. In § 2 we present the notations and basic definitions to be used. Of particular interest is the graded Lie algebra structure introduced by Schouten (1954) and Nijenhuis (1955) (see e.g. Lichnerowicz 1974) which will provide us with a method to prove in a straightforward way some of the theorems arising in the following sections. In § 3 we consider a Hamiltonian formulation of the results obtained by Hojman and Harleston (1981) in a Lagrangian formalism: for any pair of closed admissible two-forms, there are n constants of motion (of non-Noether type); some can be trivial. The coordinate-free way we use to express these results displays their generality; the explicit expression with coordinates is also given. In § 4 we come back to the Lagrange formalism to reformulate, in an intrinsic way, the results of the theorem by Hojman and Harleston (see also Henneaux 1981) in a slightly generalised way: instead of equivalent Lagrangians, only two closed admissible two-forms are needed. The terminology introduced in this section will be used in § 5 to relate these (non-Noether) constants of motion to those found by Lutzky (1979) according to an idea of Currie and Saletan (1966). Such a relation makes use of some geometrical ideas recently suggested by Giandolfi *et al* (1981). Finally we remark how this global approach furnishes more general results than the usual ones because a globally defined Lagrangian function is not needed but only a locally defined one.

2. Notations and basic definitions

A symplectic manifold of dimension $2n$ is a pair (M^{2n}, ω) where M is a C^∞-differentiable manifold of dimension $2n$ and ω a non-degenerate (maximal rank) closed two-form. By T_xM and T_x^*M we denote the tangent and cotangent spaces in $x \in M$, and by $T_q^p(M)$, $(p, q \in \mathbb{N})$, the tensor fields of type (p, q). The non-degenerate two-form ω furnishes isomorphisms $\hat{\omega}(x): T_xM \to T_x^*M$ defined by $\hat{\omega}(x)(v) = \iota(v)\omega(x)$, $v \in T_xM$, where $\iota(v)$ denotes contraction with the vector v; this allows us to identify, pointwise, vector fields on M, $\mathfrak{X}(M)$, and one-forms on M, $\Lambda^1(M)$, by means of $\hat{\omega}$. The isomorphism can be extended by tensorialisation to a new isomorphism also denoted $\hat{\omega}$, between $T_0^p(M)$ and $T_p^0(M)$. The set of antisymmetric covector fields of rank k will be denoted $\Lambda^k(M)$, whereas that of antisymmetric vector fields will be written $V^k(M)$.

A vector field X on M is said to be a locally (resp. globally) Hamiltonian vector field if $\hat{\omega}(X)$ is a closed one-form, $\hat{\omega}(X) \in Z^1(M)$ (resp. exact one-form, $\hat{\omega}(X) \in B^1(M)$). For any $f \in \Lambda^0(M) = C^\infty(M)$, $\hat{\omega}^{-1}(\mathrm{d}f)$ will be written X_f. Similarly, if $\alpha \in \Lambda^k(M)$, the element $\hat{\omega}^{-1}(\alpha) \in V^k(M)$ will be denoted X_α. Then, if X is a globally Hamiltonian vector field, there is $f \in \Lambda^0(M)$ such that $X = X_f$.

By a Hamiltonian dynamical system, we mean a triplet (M, ω, X) where (M, ω) is a symplectic manifold and X a globally Hamiltonian vector field on M, i.e. there is a function $H \in \Lambda^0(M)$, called the Hamiltonian function, such that $X = X_H$. The Hamiltonian dynamical system is sometimes written (M, ω, H). The Hamiltonian function is also called the dynamics of the system.

On $T(M) = \oplus_{k>0} T_0^k(M)$, a graded Lie algebra structure can be defined as follows. If $X \in T_0^p(M)$ and $Y \in T_0^q(M)$, $[X, Y] \in T_0^{p+q-1}(M)$ is defined as

$$\iota([X, Y])\alpha = (-1)^{pq+p}\iota(X)\,\mathrm{d}\iota(Y)\alpha + (-1)^p\iota(Y)\,\mathrm{d}\iota(X)\alpha \tag{2.1}$$

for any $\alpha \in \Lambda(M)$. Notice that the degree of elements in $T_0^p(M)$ is $p-1$, and that if $X, Y \in T_0^1(M)$ then $[X, Y]$ is just the usual Lie bracket of vector fields.

The Lie derivative of a vector field can be expressed in terms of the brackets (2.1) (see e.g. Lichnerowicz 1974) as

$$\iota(L_XY)\alpha = \iota(X)\,\mathrm{d}\iota(Y)\alpha - \iota(Y)\,\mathrm{d}\iota(X)\alpha = \iota([X, Y])\alpha. \tag{2.2}$$

The graded Lie algebra structure properties are also to be recalled,

$$[X, Y] = (-1)^{pq}[Y, X], \tag{2.3a}$$

$$(-1)^{pq}[[Y, Z], X] + (-1)^{qr}[[Z, X], Y] + (-1)^{pr}[[X, Y], Z] = 0, \tag{2.3b}$$

where $X \in T_0^{p+1}(M)$, $Y \in T_0^{q+1}(M)$ and $Z \in T_0^{r+1}(M)$.

The symplectic structure (M, ω) leads to the following properties which will be used in the next section. If $X_\omega = \hat{\omega}^{-1}(\omega)$, then $X_\omega \in V_2^p(M)$ is such that

$$[X_\omega, X_\omega] = 0. \tag{2.4}$$

Furthermore, for any $A \in T_0^r(M)$

$$\hat{\omega}[X_\omega, A] = d\hat{\omega}(A). \tag{2.5}$$

3. Admissible two-forms and constants of motion in a Hamiltonian formalism

A closed two-form $\omega' \in Z^2(M)$ is said to be admissible for the Hamiltonian system (M, ω, H) if ω' is invariant under X_H, i.e. $L_{X_H}\omega' = 0$. Therefore, $\iota(X_H)\omega' \in Z^1(M)$.

In the Hamiltonian formalism, a theorem corresponding to that of Hojman and Harleston (1981) can be stated with this language, as follows.

Theorem 1. Let ω_α, $\alpha = 1, 2$ be two closed two-forms. If ω_1 is a non-degenerate two-form and they are both admissible for the dynamical system (M, ω, X_H), then $\omega_2(X_{\omega_1})$ is a constant of motion.

Proof. The Lie derivative $L_{X_H}\omega_2(X_{\omega_1})$ is given by

$$L_{X_H}\omega_2(X_{\omega_1}) = L_{X_H}\iota(X_{\omega_1})\omega_2 = (X_{\omega_1})L_{X_H}\omega_2 + \iota([X_{\omega_1}, X_H])\omega_2.$$

As ω_2 is an admissible closed two-form, $L_{X_H}\omega_2 = 0$. On the other hand, we can use (2.5) to see that $[X_{\omega_1}, X_H] = 0$, and therefore $L_{X_H}\omega_2(X_{\omega_1}) = 0$. In fact, from (2.5) we obtain

$$\hat{\omega}_1([X_{\omega_1}, X_H]) = d\omega_1(X_H) = d(\iota(X_H)\omega_1) = 0.$$

Notice that the results of the former theorem mean that the trace of the tensor of type (2, 2), $X_{\omega_1} \otimes \omega_2$, is a constant of motion. The generalisation of this result is straightforward.

Theorem 2. With the same hypotheses as theorem 1, the trace of the tensor of type $(2k, 2k)$, $k \leqslant n$, given by

$$\rho_k = (X_{\omega_1} \otimes \omega_2) \otimes \overset{k \text{ times}}{\cdots} \otimes (X_{\omega_1} \otimes \omega_2) \tag{3.1}$$

is also a constant of motion.

Proof. Let C_i^j denote the contraction between the *i*th covariant and *j*th contravariant indices; then the Lie derivative of $\operatorname{Tr}\rho_k \equiv C_1^2 C_2^3 \cdots C_k^1 \rho_k$ is given by

$$L_{X_H} \operatorname{Tr}\rho_k = C_1^2 \cdots C_k^1 L_{X_H}(X_{\omega_1} \otimes \omega_2) \otimes \cdots \otimes (X_{\omega_1} \otimes \omega_2).$$

If we made use of $L_{X_H}\omega_\alpha = 0$ as well as $L_{X_H}X_{\omega_1} = 0$, which may be seen to be true because from (2.2)

$$L_{X_H}X_{\omega_1} = [X_H, X_{\omega_1}],$$

we obtain that $L_{X_H} \operatorname{Tr}\rho_k = 0$.

Now physicists are used to writing explicit expressions of the constants of motion in a coordinate-dependent way, namely, by using local charts; as ω_1 is non-degenerate, there is a symplectic atlas such that we can locally write ω_1 in an appropriate chart (U, ϕ), as $\omega_{1|U} = \mathrm{d}p_i \wedge \mathrm{d}q^i$ where repeated indices summation is understood. The local expression of ω_2 in the same chart will be $\omega_{2|U} = a^{ij}\, \mathrm{d}p_i \wedge \mathrm{d}p_j + b^{ij}\, \mathrm{d}p_i \wedge \mathrm{d}q^j + c^{ij}\, \mathrm{d}q^i \wedge \mathrm{d}q^j$, or in a matrix form

$$\{\omega_{2|U}\} = \left[\begin{array}{c|c} a^{ij} & b^{ij} \\ \hline -b^{ji} & c^{ij} \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline -B^t & C \end{array}\right]. \tag{3.2}$$

Moreover, the two-vector X_{ω_1} is given by

$$X_{\omega_1|U} = -(\partial/\partial p_i) \wedge \partial/\partial q^i$$

which we will write in matrix form

$$\{X_{\omega_1|U}\} = \left[\begin{array}{c|c} 0 & -\mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right]. \tag{3.3}$$

The contraction of (3.2) and (3.3) leads to

$$\omega_2(X_{\omega_1})_{|U} = 2\,\mathrm{Tr}(b^{ij}) = 2b^{ii} \tag{3.4}$$

which are the constants referred to in theorem 1 and which correspond to those found by Hojman and Harleston in a Lagrangian formalism. The corresponding constants of theorem 2 are similarly found to be

$$\mathrm{Tr}\,\rho_{k|U} = 2\,\mathrm{Tr}\{B^k\}. \tag{3.5}$$

When the matrices A or C vanish.

4. The Lagrangian formalism: Hojman and Harleston theorem

In the Lagrangian formalism the carrier space for the dynamics is the tangent bundle TQ of a configuration space Q which is a differentiable manifold. The Lagrangian is a function $L \in \Lambda^0(TQ)$ which will be assumed to be regular, i.e. the Legendre transformation, $D_L : TQ \to T^*Q$, is to be a local diffeomorphism. We recall that the map D_L is defined as $D_L(q, v) = (q, \mathrm{d}L_q(q, v))$, where $L_q : T_qQ \to R$ is given by the restriction $L_q = L_{|T_qQ}$. The natural symplectic structure on the cotangent bundle T^*Q can be transported to TQ. The canonical two-form ω_0 in T^*Q is pulled back to a non-degenerate closed two-form $\omega_L = D_L^*\omega_0$, on TQ. The local inverse of D_L gives a function $H \in \Lambda^0(T^*Q)$ which permits us to define a dynamics in T^*Q, i.e. a vector field X_H. The (locally) corresponding vector field on TQ will be denoted X_E, and therefore $D_{L*}X_E = X_H$. The vector field X_E is globally Hamiltonian; if the Lagrangian action A is defined as a function $A : TQ \to R$, given by $A(q, v) = D_L(q, v)(v)$, the function defining the vector field X_E by $\iota(X_E)\omega_L = \mathrm{d}E$ is $E = A - L$. The triplet (TQ, ω_L, X_E) is called a regular Lagrangian system. More information can be found in Abraham and Marsden (1978).

With this notation we are able to state the concept of equivalent Lagrangians: two Lagrangian functions L_α $(\alpha = 1, 2)$ are said to be equivalent if they give rise to the same dynamical vector field on TQ, namely if $X_{E_1} = X_{E_2}$. In this case $L_{X_E}\omega_{L_\alpha} = 0$, $\alpha = 1, 2$.

Given a dynamical vector field on TQ, X, a closed two-form is said to be an admissible two-form if $\iota(X)\omega$ is a closed one-form. Then, if two Lagrangian functions are equivalent, the dynamical corresponding closed two-forms ω_{L_α} are admissible two-forms for the same dynamical vector field.

The theorem by Hojman and Harleston (HH) was stated for the case of equivalent Lagrangian functions. It can also be formulated in a more general case and in an intrinsic coordinate-free way, as follows.

Theorem 3. Let (TQ, ω_L, X_E) be a regular Lagrangian dynamical system. If ω_α, $\alpha = 1, 2$, are two admissible closed two-forms for the dynamics X_E and ω_1 is non-degenerate, then the trace of the tensor $\rho_k = (X_{\omega_1} \otimes \omega_2)^{\otimes k}$ is a constant of motion for any positive integer number k.

The results of the HH theorem arise now as a corollary.

Corollary. Given two equivalent Lagrangians L_1, L_2, and if L_1 is regular, then the trace of the tensor

$$\rho_k = (X_{\omega_{L_1}} \otimes \omega_{L_2}) \otimes \overset{k \text{ times}}{\cdots} \otimes (X_{\omega_{L_1}} \otimes \omega_{L_2})$$

is a constant of motion, for any positive integer number k.

The proof of the theorem follows the pattern of that of theorem 2 and will not be given. Instead we will express the two-forms ω_{L_α} arising in the corollary in particular coordinates in order to recover the results of the HH theorem. Both two-forms are expressed in a coordinate chart

$$\omega_{L_\alpha|U} = \frac{\partial^2 L_\alpha}{\partial \dot{q}^k \, \partial \dot{q}^j} \, d\dot{q}^k \wedge dq^j + \frac{\partial^2 L_\alpha}{\partial \dot{q}^k \, \partial q^j} \, dq^k \wedge dq^j, \qquad \alpha = 1, 2, \tag{4.1}$$

while

$$X_{\omega_{L_1}|U} = \{A_1^{-1}\}_{ij} (\partial/\partial \dot{q}^i) \wedge \partial/\partial q^j$$

where $\{A_\alpha\}_{kj} = \partial^2 L_\alpha / \partial \dot{q}^k \, \partial \dot{q}^j$. Therefore $\mathrm{Tr}\, \rho_1 = \mathrm{Tr}(X_{\omega_{L_1}} \otimes \omega_{L_2}) = \mathrm{Tr}\{A_1^{-1} A_2\}$. A similar expression holds for $k > 1$, namely $\mathrm{Tr}\, \rho_k = \mathrm{Tr}\{(A_1^{-1} A_2)^k\}$. This is the way in which HH presented their results.

5. Non-Noether constants of motion

It was recently shown that if L_1 and L_2 are two equivalent Lagrangians, the quotient $\det A_1 / \det A_2$ is a constant of the motion (Lutzky 1979, Giandolfi *et al* 1981). This result generalises that of Currie and Saletan (1966) for the particular case $n = 1$. The proof given by Giandolfi *et al* (1981) is particularly simple and beautiful: we feel the convenience of spending a little time in summing up such a derivation because it leads in a natural way to the concept of a pencil of admissible Lagrangian functions which will permit us to compare non-Noether constants of motion found by Giandolfi *et al* with those of the Hojman and Harleston theorem.

Let L_1 be a regular Lagrangian; the closed two-form ω_{L_1} is a nondegenerate and $(\omega_{L_1})^{\wedge n}$ is a basis for the $C^\infty(TQ)$-module $\Lambda^{2n}(TQ)$. If L_2 is a Lagrangian function equivalent to an L_1, then $(\omega_{L_2})^{\wedge n} = f_0 (\omega_{L_1})^{\wedge n}$ and it is quite easy to check that f_0 is a constant of motion; on a coordinate chart such a constant is but the quotient of the determinant quoted above. Moreover, Giandolfi *et al* proposed new constants of motion f_k given by the relation

$$(\omega_{L_2})^{\wedge (n-k)} \wedge (\omega_{L_1})^{\wedge k} = f_k (\omega_{L_1})^{\wedge n} \tag{5.1}$$

which they called non-Noether constants of motion. Notice that on the left-hand side arise the coefficients of the different powers of λ in $(\omega_{L_2}-\lambda\omega_{L_1})^{\Lambda n}$. This fact suggests the introduction of the following definitions.

Let (TQ, ω_L, X_E) be a regular Lagrangian system, and ω_1 and ω_2 admissible closed two-forms for X_E, ω_1 being regular. The pencil of admissible closed two-forms defined by them is the set

$$\{\omega_2-\lambda\omega_1|\lambda\in R\}.$$

The real function $f: R\times TQ\to R$ defined by

$$(\omega_2-\lambda\omega_1)^{\hat{n}}=f(\lambda)\omega_1{}^{\hat{n}} \tag{5.2}$$

is called the characteristic function of the pencil.

In particular, if L' is a Lagrange function equivalent to L, ω_L and $\omega_{L'}$ define the pencil of admissible closed two-forms for the common dynamical vector field.

$$\omega_{(L'-\lambda L)}=\omega_{L'}-\lambda\omega_L.$$

Theorem 4. The characteristic function of the pencil of admissible closed two-forms, $\omega_2-\lambda\omega_1$, for the regular Lagrangian system (TQ, ω_L, X_E) is a constant of the motion.

Proof. As ω_α $(\alpha=1,2)$ are admissible closed two-forms, $L_{X_E}\omega_\alpha=0$, and consequently $L_{X_E}(\omega_2-\lambda\omega_1)=0$. When we identify the Lie derivatives of both sides of equation (5.2) we find that $L_{X_E}f(\lambda)=0$.

Notice that when comparing (5.1) and (5.2) for $\omega_\alpha=\omega_{L_\alpha}$ we see that $f(\lambda)=\Sigma_{k=0}^{n}\lambda^k f^k$.

Moreover, when the admissible closed two-forms ω_α defining the pencils are defined by two Lagrange functions L_α $(\alpha=1,2)$, the local expressions in coordinates are as in (4.1), i.e. the matrices associated with the two-forms ω_α are

$$\omega_{L_\alpha|U}=\left(\begin{array}{c|c}0 & A_\alpha\\ \hline -A_\alpha & B_\alpha\end{array}\right)$$

where $\{B_\alpha\}_{jk}=\partial^2L_\alpha/\partial\dot{q}^j\,\partial q^k-\partial^2L_\alpha/\partial\dot{q}^k\,\partial q^j$; $\{A_\alpha\}_{jk}=\partial^2L_\alpha/\partial\dot{q}^j\,\partial\dot{q}^k$.

Theorem 5. Let (TQ, ω_L, X_E) be a regular Lagrangian system and L_1, L_2 two Larangian functions such that ω_{L_1}, ω_{L_2} are admissible closed two-forms for X_E, L_1 being regular. The characteristic function of the pencil in a chart is given by $f(\lambda)_{|U}=\det(A_2A_1^{-1}-\lambda)$ with A as in the preceding theorem.

Proof. The expressions in coordinates of ω_{L_1}, ω_{L_2} and $\omega_{L_1}-\lambda\omega_{L_2}$ lead to

$$(\omega_{L_\alpha|U})^{\Lambda n}=\det(A_\alpha)\,d\dot{q}^1\wedge d\dot{q}^2\wedge\ldots\wedge d\dot{q}^n\wedge dq^1\ldots\wedge dq^n,$$
$$(\omega_{(L_2-\lambda L_1)})^{\Lambda n}=\det(A_2-\lambda A_1)\,d\dot{q}^1\wedge d\dot{q}^2\ldots\wedge d\dot{q}^n\wedge dq^1\ldots\wedge dq^n,$$

and therefore

$$(\omega_{L_2}-\lambda\omega_{L_1})^{\Lambda n}=\det(A_2-\lambda A_1)\det(A_1^{-1})(\det(A_1)\,d\dot{q}^1\wedge\ldots\wedge d\dot{q}^n\wedge dq^1\ldots\wedge dq^n).$$

Then the characteristic function $f(\lambda)$ in the chart considered is given by $f(\lambda)_{|U}=\det(A_2A_1^{-1}-\lambda)=\det(A_1^{-1}A_2-\lambda)$.

The results of the theorem include those of Hojman and Harleston, as we can realise by making use of the Le Verrier method of determining the characteristic equation of a matrix (see e.g. Wilkinson 1965): the coefficients c_k of the characteristic equation of the matrix M are related to the traces of increasing powers M^k, σ_k, by means of Newton's equations:

$$c_1 = -\sigma_1, \qquad kc_k = -(\sigma_k + c_1\sigma_{k-1} + \ldots + c_{k-1}\sigma_1), \qquad k > 2.$$

In the particular case of M being the matrix $A_1^{-1}A_2$, we will find the following relation between the Hojman–Harleston constants of motion $p_k = \operatorname{Tr} \rho_k$ and those of theorem 5.

$$f_1 = -p_1, \qquad kf_k = -(p_k + f_1 p_{k-1} + \ldots + f_{k-1}p_1).$$

Before ending this paper, we would like to remark that the above results hold even if there are no global Lagrangians L_α, but both of them are locally Lagrangian, i.e. ω_α are exact forms, $\omega_\alpha = \mathrm{d}\theta_\alpha$, such that $\dot{\mathrm{d}}\theta_\alpha = 0$, where d denotes vertical derivative (Godbillon 1969): in that case, for any $m \in TQ$ there is a neighbourhood U of m and $C^\infty(U)$ functions L_α such that $\omega_{\alpha|U} = \omega_{L_\alpha} = \mathrm{d}(\dot{\mathrm{d}}L_\alpha)$.

Acknowledgment

We thank the Instituto de Estudios Nucleares for partial financial support. One of the authors (LAI) is also indebted to the Direction General de Política Científica for a grant. Interesting remarks made by Professor G Marmo are also acknowledged.

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